Contents 14

Applications of Integration 1

Learning outcomes

In this Workbook you will learn to interpret an integral as the limit of a sum. You will learn how to apply this approach to the meaning of an integral to calculate important attributes of a curve: the area under the curve, the length of a curve segment, the volume and surface area obtained when a segment of a curve is rotated about an axis. Other quantities of interest which can also be calculated using integration is the position of the centre of mass of a plane lamina and the moment of inertia of a lamina about an axis. You will also learn how to determine the mean value of an integal.

Integration as the Limit of a Sum

14.1

Introduction

In HELM 13, integration was introduced as the reverse of differentiation. A more rigorous treatment would show that integration is a process of adding or 'summation'. By viewing integration from this perspective it is possible to apply the techniques of integration to finding areas, volumes, centres of gravity and many other important quantities.

The content of this Section is important because it is here that integration is defined more carefully. A thorough understanding of the process involved is essential if you need to apply integration techniques to practical problems.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals

Learning Outcomes

On completion you should be able to ...

- explain integration as the limit of a sum
- evaluate the limit of a sum in simple cases

1. The limit of a sum

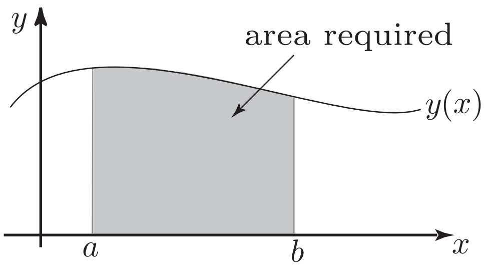
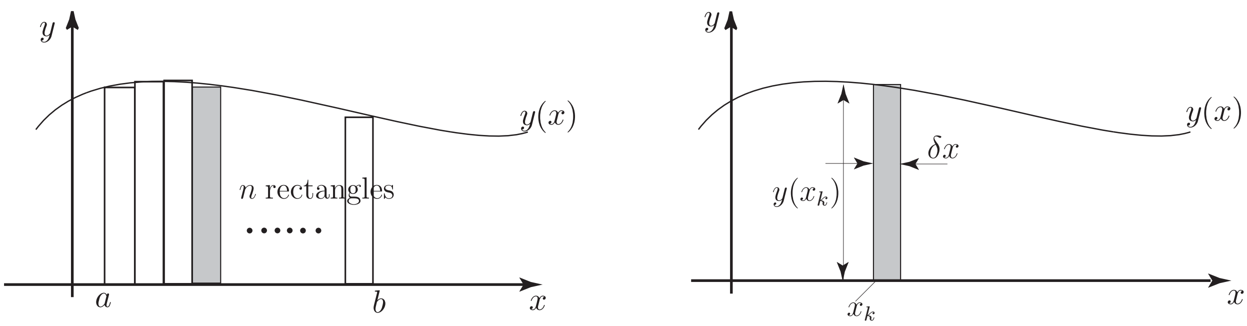


Figure 1: The area under a curve

Consider the graph of the positive function $y(x)$ shown in Figure 1. Suppose we are interested in finding the area under the graph between $x = a$ and $x = b$. One way in which this area can be **approximated** is to divide it into a number of rectangles of equal width, find the area of each rectangle, and then add up all these individual rectangular areas. This is illustrated in Figure 2a, which shows the area divided into n rectangles (with some small discrepancies at the tops), and Figure 2b which shows the dimensions of a typical rectangle which is located at $x = x_k$.



(a) The area approximated by n rectangles

(b) A typical rectangle

Figure 2

We wish to find an expression for the area under a curve based on the sum of many rectangles. Firstly, we note that the distance from $x = a$ to $x = b$ is $b - a$. In Figure 2a the area has been divided into n rectangles. If n rectangles span the distance from a to b the width of each rectangle is $\dfrac{b-a}{n}$:

It is conventional to label the width of each rectangle as δx, i.e. $\delta x = \dfrac{b-a}{n}$. We label the x coordinates at the left-hand side of the rectangles as x_1, x_2 up to x_n (here $x_1 = a$ and $x_{n+1} = b$). A typical rectangle, the kth rectangle, is shown in Figure 2b. Note that its height is $y(x_k)$, so its area is $y(x_k) \times \delta x$.

The sum of the areas of all n rectangles is then

$$y(x_1)\delta x + y(x_2)\delta x + y(x_3)\delta x + \cdots + y(x_n)\delta x$$

which we write concisely using sigma notation as

$$\sum_{k=1}^{n} y(x_k)\delta x$$

This quantity gives us an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. So, what we want to find is the value of this sum when n tends to infinity and δx tends to zero. We write this value as

$$\lim_{n\to\infty}\sum_{k=1}^{n} y(x_k)\delta x$$

The lower and upper limits on the sum correspond to the first rectangle and last rectangle where $x = a$ and $x = b$ respectively and so we can write this limit in the equivalent form

$$\lim_{\delta x\to 0}\sum_{x=a}^{x=b} y(x)\delta x \qquad (1)$$

Here, as the number of rectangles increases without bound we drop the subscript k from x_k and write $y(x)$ which is the value of y at a 'typical' value of x. If this sum can actually be found, it is called the **definite integral** of $y(x)$, from $x = a$ to $x = b$ and it is written $\int_a^b y(x)dx$. You are already familiar with the technique for evaluating definite integrals which was studied in Section 14.2. Therefore we have the following definition:

Key Point 1

The definite integral $\int_a^b y(x)dx$ is defined as $\lim_{\delta x\to 0}\sum_{x=a}^{x=b} y(x)\delta x$

Note that the quantity δx represents the thickness of a small but finite rectangle. When we have taken the limit as δx tends to zero to obtain the integral, we write dx, which reminds us of the variable of integration.

This process of dividing an area into very small regions, performing a calculation on each region, and then adding the results by means of an integral is very important. This will become apparent when finding volumes, centres of gravity, moments of inertia etc in the following Sections where similar procedures are followed.

Example 1

The area under the graph of $y = x^2$ between $x = 0$ and $x = 1$ is to be found by approximating it by a large number of thin rectangles and finding the limit of the sum of their areas. From Equation (1) this is $\lim_{\delta x \to 0} \sum_{x=0}^{x=1} y(x)\,\delta x$. Write down the integral which this sum defines and evaluate it to obtain the area under the curve.

Solution

The limit of the sum defines the integral $\int_0^1 y(x)dx$. Here $y = x^2$ and so $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

To show that the process of taking the limit of a sum actually works we investigate the problem in detail. We use the idea of the limit of a sum to find the area under the graph of $y = x^2$ between $x = 0$ and $x = 1$, as illustrated in Figure 3.

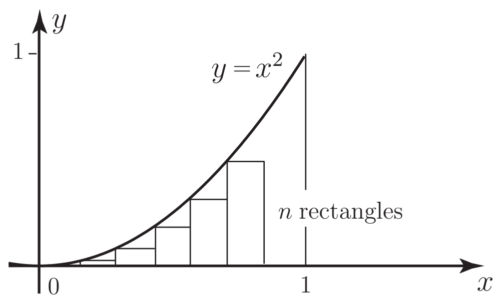
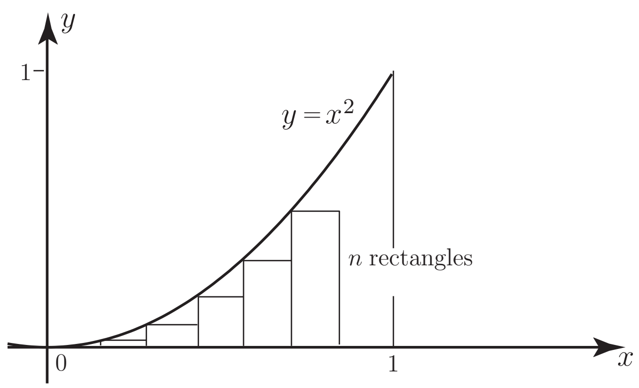


Figure 3: The area under $y = x^2$ is approximated by a number of thin rectangles

Task

Refer to the diagram below to help you answer the questions below.



If the interval between $x = 0$ and $x = 1$ is divided into n rectangles what is the width of each rectangle?

Your solution

Answer

$1/n$

Mark this on the diagram. What is the x coordinate at the left-hand side of the first rectangle ?

Your solution

Answer

0

What is the x coordinate at the left-hand side of the second rectangle ?

Your solution

Answer

$1/n$

What is the x coordinate at the left-hand side of the third rectangle ?

Your solution

Answer

$2/n$

Mark these coordinates on the diagram.
What is the x coordinate at the left-hand side of the kth rectangle ?

Your solution

Answer

$(k-1)/n$

Given that $y = x^2$, what is the y coordinate at the left-hand side of the kth rectangle ?

Your solution

Answer

$$\left(\frac{k-1}{n}\right)^2$$

The area of the kth rectangle is its height $\times$ its width. Write down the area of the kth rectangle:

Your solution

Answer

$$\left(\frac{k-1}{n}\right)^2 \times \frac{1}{n} = \frac{(k-1)^2}{n^3}$$

To find the total area A_n of the n rectangles we must add up all these individual rectangular areas:

$$A_n = \sum_{k=1}^{n} \frac{(k-1)^2}{n^3}$$

This sum can be simplified and then calculated as follows. You will need to make use of the formulas for the sum of the first n integers, and the sum of the squares of the first n integers:

$$\sum_{k=1}^{n} 1 = n, \qquad \sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Then, the total area of the rectangles is given by

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} \\
&= \frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2 \\
&= \frac{1}{n^3}\sum_{k=1}^{n}(k^2-2k+1) \\
&= \frac{1}{n^3}\left(\sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right) \\
&= \frac{1}{n^3}\left(\frac{n}{6}(n+1)(2n+1) - 2\frac{n}{2}(n+1) + n\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - (n+1) + 1\right) \\
&= \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6} - n\right) \\
&= \frac{1}{6n^2}\left(2n^2-3n+1\right) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}
\end{aligned}$$

Note that this is a formula for the **exact** total area of the n rectangles. It is an **estimate** of the area under the graph of $y = x^2$. However, as n gets larger, the terms $\frac{1}{2n}$ and $\frac{1}{6n^2}$ become small and will eventually tend to zero. If we let n tend to infinity we obtain the exact answer of $\frac{1}{3}$.

The required area is $\frac{1}{3}$. It has been found as **the limit of a sum** and of course agrees with that calculated by integration.

In the calculations which follow in subsequent Sections the need to evaluate complicated limits like this is avoided by performing the integration using the techniques of HELM 13. Nevertheless it will sometimes be necessary to go through the process of dividing a region into small sections, performing a calculation on each section and then adding the results, in order to formulate the integral required. When numerical methods of integration are studied (HELM 31) this summation method will prove fundamental.

Engineering Example 1

Pulley belt tension

Problem

Consider that a belt is partially wound around a pulley so that there is a difference in the tension either side of the pulley (see Figure 4). The pulley will be stationary as long as the friction between belt and pulley is sufficient. The frictional force on the pulley will depend on the extent of the contact between belt and pulley i.e. on the angle θ shown in Figure 4. Given that the tensions on either side of the belt are T_2 and T_1 and that the coefficient of friction between belt and pulley is μ, find an expression for T_2 in terms of T_1, μ and θ.

Solution

Consider a small element of the belt, at angle θ where the tension is T. Changing the angle by a small amount $\Delta\theta$ changes the tension from T to $T + \Delta T$.

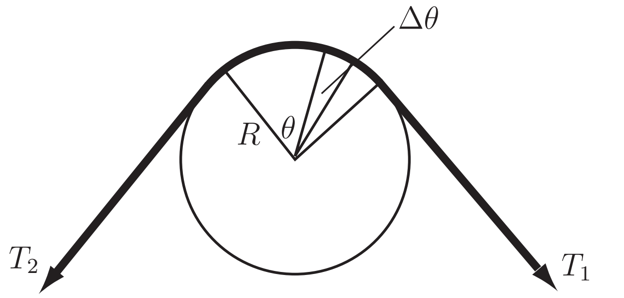


Figure 4

Take moments about the centre of the pulley, denoting the radius of the pulley by R and assuming that the frictional force is μT per unit length. For the pulley to remain stationary,

$$R\Delta\theta\mu T = R(T + \Delta T) - RT \quad \text{or} \quad \Delta\theta = \frac{\Delta T}{\mu T}.$$

Using integration as the limit of a sum,

$$\theta = \int_{T_1}^{T_2} \frac{dT}{\mu T} = \frac{1}{\mu}\Big[\ln T\Big]_{T_1}^{T_2} = \frac{1}{\mu}\ln\left(\frac{T_2}{T_1}\right). \quad \text{So } T_2 = T_1 e^{\mu\theta}.$$

Exercises

1. Find the area under $y = x + 1$ from $x = 0$ to $x = 10$ using the limit of a sum.

2. Find the area under $y = 3x^2$ from $x = 0$ to $x = 2$ using the limit of a sum.

3. Write down, but do not evaluate, the integral defined by the limit as $\delta x \to 0$, or $\delta t \to 0$ of the following sums:

   (a) $\displaystyle\sum_{x=0}^{x=1} x^3 \delta x$, (b) $\displaystyle\sum_{x=0}^{x=4} 4\pi x^2 \delta x$, (c) $\displaystyle\sum_{t=0}^{t=1} t^3 \delta t$, (d) $\displaystyle\sum_{x=0}^{x=1} 6mx^2 \delta x$.

Answers

1. 60,

2. 8,

3. (a) $\displaystyle\int_0^1 x^3 dx$, (b) $4\pi \displaystyle\int_0^4 x^2 dx$, (c) $\displaystyle\int_0^1 t^3 dt$, (d) $6m \displaystyle\int_0^1 x^2 dx$.

The Mean Value and the Root-Mean-Square Value

14.2

Introduction

Currents and voltages often vary with time and engineers may wish to know the mean value of such a current or voltage over some particular time interval. The mean value of a time-varying function is defined in terms of an integral. An associated quantity is the **root-mean-square** (r.m.s). For example, the r.m.s. value of a current is used in the calculation of the power dissipated by a resistor.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals
- be familiar with a table of trigonometric identities

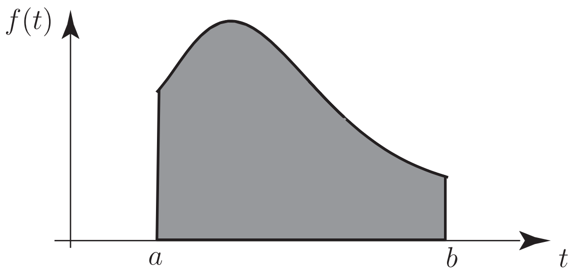
Learning Outcomes

On completion you should be able to ...

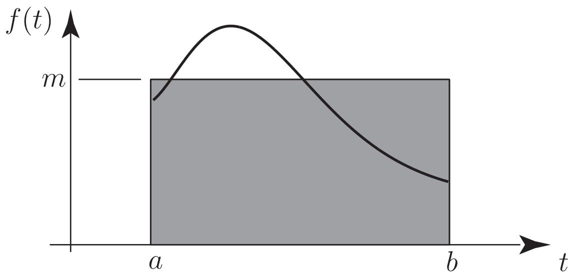
- calculate the mean value of a function
- calculate the root-mean-square value of a function

1. Average value of a function

Suppose a time-varying function $f(t)$ is defined on the interval $a \leq t \leq b$. The area, A, under the graph of $f(t)$ is given by the integral $A = \int_a^b f(t)\,dt$. This is illustrated in Figure 5.



(a) the area under the curve from $t = a$ to $t = b$



(b) the area under the curve and the area of the rectangle are equal

Figure 5

On Figure 3 we have also drawn a rectangle with base spanning the interval $a \leq t \leq b$ and which has the same area as that under the curve. Suppose the height of the rectangle is m. Then

$$\text{area of rectangle} = \text{area under curve} \quad \Rightarrow \quad m(b-a) = \int_a^b f(t)\,dt \quad \Rightarrow \quad m = \frac{1}{b-a}\int_a^b f(t)\,dt$$

The value of m is the **mean value** of the function across the interval $a \leq t \leq b$.

Key Point 2

The **mean value** of a function $f(t)$ in the interval $a \leq t \leq b$ is $\displaystyle\frac{1}{b-a}\int_a^b f(t)\,dt$

The mean value depends upon the interval chosen. If the values of a or b are changed, then the mean value of the function across the interval from a to b will in general change as well.

Example 2

Find the mean value of $f(t) = t^2$ over the interval $1 \leq t \leq 3$.

Solution

Using Key Point 2 with $a = 1$ and $b = 3$ and $f(t) = t^2$

$$\text{mean value} = \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{3-1}\int_1^3 t^2\,dt = \frac{1}{2}\left[\frac{t^3}{3}\right]_1^3 = \frac{13}{3}$$

Find the mean value of $f(t) = t^2$ over the interval $2 \leq t \leq 5$.

Use Key Point 2 with $a = 2$ and $b = 5$ to write down the required integral:

Your solution

mean value =

Answer

$$\frac{1}{5-2}\int_2^5 t^2\,dt$$

Now evaluate the integral:

Your solution

mean value =

Answer

$$\frac{1}{5-2}\int_2^5 t^2\,dt = \frac{1}{3}\left[\frac{t^3}{3}\right]_2^5 = \frac{1}{3}\left[\frac{125}{3} - \frac{8}{3}\right] = \frac{117}{9} = 13$$

Engineering Example 2

Sonic boom

Introduction

Impulsive signals are described by their peak amplitudes and their duration. Another quantity of interest is the total energy of the impulse. The effect of a blast wave from an explosion on structures, for example, is related to its total energy. This Example looks at the calculation of the energy on a sonic boom. Sonic booms are caused when an aircraft travels faster than the speed of sound in air. An idealized sonic-boom pressure waveform is shown in Figure 6 where the instantaneous sound pressure $p(t)$ is plotted versus time t. This wave type is often called an N-wave because it resembles the shape of the letter N. The energy in a sound wave is proportional to the square of the sound pressure.

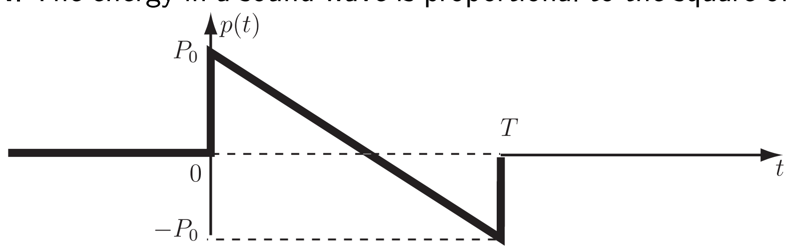


Figure 6: An idealized sonic-boom pressure waveform

Problem in words

Calculate the energy in an ideal N-wave sonic boom in terms of its peak pressure, its duration and the density and sound speed in air.

Mathematical statement of problem

Represent the positive peak pressure by P_0 and the duration by T. The total acoustic energy E carried across unit area normal to the sonic-boom wave front during time T is defined by

$$E = <p(t)^2> T/\rho c \qquad (1)$$

where ρ is the air density, c the speed of sound and the time average of $[p(t)]^2$ is

$$<p(t)^2> = \frac{1}{T}\int_0^T p(t)^2\,dt \qquad (2)$$

(a) Find an appropriate expression for $p(t)$.

(b) Hence show that E can be expressed in terms of $P_0,\ T,\ \rho$ and c as $E = \dfrac{TP_0^2}{3\rho c}$.

Mathematical analysis

(a) The interval of integration needed to compute (2) is $[0, T]$. Therefore it is necessary to find an expression for $p(t)$ only in this interval. Figure 6 shows that, in this interval, the dependence of the sound pressure p on the variable t is linear, i.e. $p(t) = at + b$.

From Figure 6 also $p(0) = P_0$ and $p(T) = -P_0$. The constants a and b are determined from these conditions.

At $t = 0,\ a \times 0 + b = P_0$ implies that $b = P_0$.

At $t = T,\ a \times T + b = -P_0$ implies that $a = -2P_0/T$.

Consequently, the sound pressure in the interval $[0, T]$ may be written $p(t) = \dfrac{-2P_0}{T}\,t + P_0$.

(b) This expression for $p(t)$ may be used to compute the integral (2)

$$\begin{aligned}
\frac{1}{T}\int_0^T p(t)^2\,dt &= \frac{1}{T}\int_0^T \left(\frac{-2P_0}{T}\,t + P_0\right)^2 dt = \frac{1}{T}\int_0^T \left(\frac{4P_0^2}{T^2}\,t^2 - \frac{4P_0^2}{T}\,t + P_0^2\right) dt \\
&= \frac{1}{T}\left[\frac{4P_0^2}{3T^2}\,t^3 - \frac{2P_0^2}{T}\,t^2 + P_0^2 t\right]_0^T \\
&= \frac{P_0^2}{T}\left(\frac{4}{3T^2}\,T^3 - \frac{2}{T}\,T^2 + T\right) - 0 = P_0^2/3.
\end{aligned}$$

Hence, from Equation (1), the total acoustic energy E carried across unit area normal to the sonic-boom wave front during time T is $E = \dfrac{TP_0^2}{3\rho c}$.

Interpretation

The energy in an N-wave is given by a third of the sound intensity corresponding to the peak pressure multiplied by the duration.

Exercises

1. Calculate the mean value of the given functions across the specified interval.

   (a) $f(t) = 1 + t$ across $[0, 2]$

   (b) $f(x) = 2x - 1$ across $[-1, 1]$

   (c) $f(t) = t^2$ across $[0, 1]$

   (d) $f(t) = t^2$ across $[0, 2]$

   (e) $f(z) = z^2 + z$ across $[1, 3]$

2. Calculate the mean value of the given functions over the specified interval.

   (a) $f(x) = x^3$ across $[1, 3]$

   (b) $f(x) = \dfrac{1}{x}$ across $[1, 2]$

   (c) $f(t) = \sqrt{t}$ across $[0, 2]$

   (d) $f(z) = z^3 - 1$ across $[-1, 1]$

3. Calculate the mean value of the following:

   (a) $f(t) = \sin t$ across $\left[0, \frac{\pi}{2}\right]$

   (b) $f(t) = \sin t$ across $[0, \pi]$

   (c) $f(t) = \sin \omega t$ across $[0, \pi]$

   (d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$

   (e) $f(t) = \cos t$ across $[0, \pi]$

   (f) $f(t) = \cos \omega t$ across $[0, \pi]$

   (g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, 1]$

4. Calculate the mean value of the following functions:

   (a) $f(t) = \sqrt{t+1}$ across $[0, 3]$

   (b) $f(t) = e^t$ across $[-1, 1]$

   (c) $f(t) = 1 + e^t$ across $[-1, 1]$

Answers

1. (a) 2 (b) -1 (c) $\dfrac{1}{3}$ (d) $\dfrac{4}{3}$ (e) $\dfrac{19}{3}$

2. (a) 10 (b) 0.6931 (c) 0.9428 (d) -1

3. (a) $\dfrac{2}{\pi}$ (b) $\dfrac{2}{\pi}$ (c) $\dfrac{1}{\pi\omega}[1 - \cos(\pi\omega)]$ (d) $\dfrac{2}{\pi}$ (e) 0 (f) $\dfrac{\sin(\pi\omega)}{\pi\omega}$

   (g) $\dfrac{1 + \sin\omega - \cos\omega}{\omega}$

4. (a) $\dfrac{14}{9}$ (b) 1.1752 (c) 2.1752

2. Root-mean-square value of a function

If $f(t)$ is defined on the interval $a \leq t \leq b$, the **mean-square** value is given by the expression:

$$\frac{1}{b-a}\int_a^b [f(t)]^2\,dt$$

This is simply the mean value of $[f(t)]^2$ over the given interval.

The related quantity: the **root-mean-square (r.m.s.)** value is given by the following formula.

Key Point 3

Root-Mean-Square Value

$$\text{r.m.s value } = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2\,dt}$$

The r.m.s. value depends upon the interval chosen. If the values of a or b are changed, then the r.m.s. value of the function across the interval from a to b will in general change as well. Note that when finding an r.m.s. value the function must be squared before it is integrated.

Example 3

Find the r.m.s. value of $f(t) = t^2$ across the interval from $t = 1$ to $t = 3$.

Solution

$$\text{r.m.s} = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2\,dt} = \sqrt{\frac{1}{3-1}\int_1^3 [t^2]^2\,dt} = \sqrt{\frac{1}{2}\int_1^3 t^4\,dt} = \sqrt{\frac{1}{2}\left[\frac{t^5}{5}\right]_1^3} \approx 4.92$$

Example 4

Calculate the r.m.s value of $f(t) = \sin t$ across the interval $0 \leq t \leq 2\pi$.

Solution

Here $a = 0$ and $b = 2\pi$ so r.m.s $= \sqrt{\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2 t\, dt}$.

The integral of $\sin^2 t$ is performed by using trigonometrical identities to rewrite it in the alternative form $\frac{1}{2}(1 - \cos 2t)$. This technique was described in HELM 13.7.

$$\text{r.m.s. value} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\cos 2t)}{2} dt} = \sqrt{\frac{1}{4\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}} = \sqrt{\frac{1}{4\pi}(2\pi)} = \sqrt{\frac{1}{2}} = 0.707$$

Thus the r.m.s value is 0.707 to 3 d.p.

In the previous Example the amplitude of the sine wave was 1, and the r.m.s. value was 0.707. In general, if the amplitude of a sine wave is A, its r.m.s value is $0.707A$.

Key Point 4

The r.m.s value of any sinusoidal waveform taken across an interval of width equal to one period is $0.707 \times$ amplitude of the waveform.

Engineering Example 3

Electrodynamic meters

Introduction

A dynamometer or electrodynamic meter is an analogue instrument that can measure d.c. current or a.c. current up to a frequency of 2 kHz. A typical dynamometer is shown in Figure 7.

It consists of a circular dynamic coil positioned in a magnetic field produced by two wound circular stator coils connected in series with each other. The torque T on the moving coil depends upon the mutual inductance between the coils given by:

$$T = I_1 I_2 \frac{dM}{d\theta}$$

where I_1 is the current in the fixed coil, I_2 the current in the moving coil and θ is the angle between the coils. The torque is therefore proportional to the square of the current. If the current is alternating the moving coil is unable to follow the current and the pointer position is related to the mean value of the square of the current. The scale can be suitably graduated so that the pointer position shows the square root of this value, i.e. the r.m.s. current.

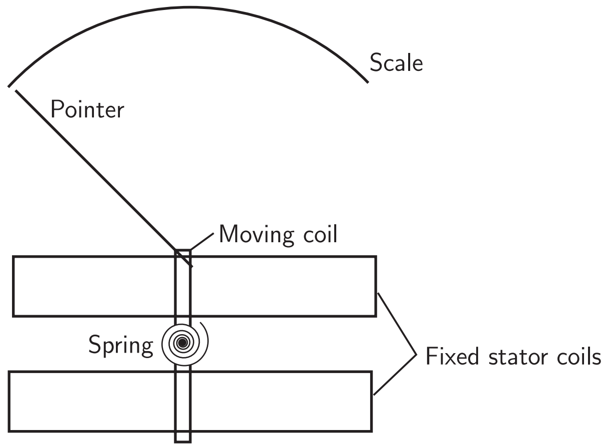


Figure 7: An electrodynamic meter

Problem in words

A dynamometer is in a circuit in series with a 400 Ω resistor, a rectifying device and a 240 V r.m.s alternating sinusoidal power supply. The rectifier resists current with a resistance of 200 Ω in one direction and a resistance of 1 kΩ in the opposite direction. Calculate the reading indicated on the meter.

Mathematical Statement of the problem

We know from Key Point 4 in the text that the r.m.s. value of any sinusoidal waveform taken across an interval equal to one period is 0.707 × amplitude of the waveform. Where 0.707 is an approximation of $\dfrac{1}{\sqrt{2}}$. This allows us to state that the amplitude of the sinusoidal power supply will be:

$$V_{\text{peak}} = \frac{V_{\text{rms}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}V_{\text{rms}}$$

In this case the r.m.s power supply is 240 V so we have

$$V_{\text{peak}} = 240 \times \sqrt{2} = 339.4\ V$$

During the part of the cycle where the voltage of the power supply is positive the rectifier behaves as a resistor with resistance of 200 Ω and this is combined with the 400 Ω resistance to give a resistance of 600 Ω in total. Using Ohm's law

$$V = IR \Rightarrow I = \frac{V}{R}$$

As $V = V_{\text{peak}} \sin(\theta)$ where $\theta = \omega t$ where ω is the angular frequency and t is time we find that during the positive part of the cycle

$$I^2_{\text{rms}} = \frac{1}{2\pi}\int_0^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta$$

During the part of the cycle where the voltage of the power supply is negative the rectifier behaves as a resistor with resistance of 1 kΩ and this is combined with the 400 Ω resistance to give 1400 Ω in total.

So we find that during the negative part of the cycle

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

Therefore over an entire cycle

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

We can calculate this value to find I_{rms}^2 and therefore I_{rms}.

Mathematical analysis

$$I_{\text{rms}}^2 = \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{339.4\sin(\theta)}{600}\right)^2 d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(\frac{339.4\sin(\theta)}{1400}\right)^2 d\theta$$

$$I_{\text{rms}}^2 = \frac{339.4^2}{2\pi\times 10000}\left(\int_{0}^{\pi}\frac{\sin^2(\theta)}{36}\,d\theta + \int_{\pi}^{2\pi}\frac{\sin^2(\theta)}{196}\,d\theta\right)$$

Substituting the trigonometric identity $\sin^2(\theta) \equiv \dfrac{1-\cos(2\theta)}{2}$ we get

$$\begin{aligned}
I_{\text{rms}}^2 &= \frac{339.4^2}{4\pi\times 10000}\left(\int_{0}^{\pi}\frac{1-\cos(2\theta)}{36}\,d\theta + \int_{\pi}^{2\pi}\frac{1-\cos(2\theta)}{196}\,d\theta\right)\\
&= \frac{339.4^2}{4\pi\times 10000}\left(\left[\frac{\theta}{36}-\frac{\sin(2\theta)}{72}\right]_0^{\pi} + \left[\frac{\theta}{196}-\frac{\sin(2\theta)}{392}\right]_{\pi}^{2\pi}\right)\\
&= \frac{339.4^2}{4\pi\times 10000}\left(\frac{\pi}{36}+\frac{\pi}{196}\right) = 0.0946875\ A^2\\
I_{\text{rms}} &= 0.31\ A \text{ to 2 d.p.}
\end{aligned}$$

Interpretation

The reading on the meter would be 0.31 A.

Exercises

1. Calculate the r.m.s values of the given functions across the specified interval.

   (a) $f(t) = 1 + t$ across $[0, 2]$

   (b) $f(x) = 2x - 1$ across $[-1, 1]$

   (c) $f(t) = t^2$ across $[0, 1]$

   (d) $f(t) = t^2$ across $[0, 2]$

   (e) $f(z) = z^2 + z$ across $[1, 3]$

2. Calculate the r.m.s values of the given functions over the specified interval.

   (a) $f(x) = x^3$ across $[1, 3]$

   (b) $f(x) = \dfrac{1}{x}$ across $[1, 2]$

   (c) $f(t) = \sqrt{t}$ across $[0, 2]$

   (d) $f(z) = z^3 - 1$ across $[-1, 1]$

3. Calculate the r.m.s values of the following:

   (a) $f(t) = \sin t$ across $\left[0, \dfrac{\pi}{2}\right]$

   (b) $f(t) = \sin t$ across $[0, \pi]$

   (c) $f(t) = \sin \omega t$ across $[0, \pi]$

   (d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$

   (e) $f(t) = \cos t$ across $[0, \pi]$

   (f) $f(t) = \cos \omega t$ across $[0, \pi]$

   (g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, 1]$

4. Calculate the r.m.s values of the following functions:

   (a) $f(t) = \sqrt{t + 1}$ across $[0, 3]$

   (b) $f(t) = e^t$ across $[-1, 1]$

   (c) $f(t) = 1 + e^t$ across $[-1, 1]$

Answers

1. (a) 2.0817 (b) 1.5275 (c) 0.4472 (d) 1.7889 (e) 6.9666
2. (a) 12.4957 (b) 0.7071 (c) 1 (d) 1.0690
3. (a) 0.7071 (b) 0.7071 (c) $\sqrt{\dfrac{1}{2} - \dfrac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$

   (d) 0.7071 (e) 0.7071 (f) $\sqrt{\dfrac{1}{2} + \dfrac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$ (g) $\sqrt{1 + \dfrac{\sin^2 \omega}{\omega}}$
4. (a) 1.5811 (b) 1.3466 (c) 2.2724

Volumes of Revolution 14.3

Introduction

In this Section we show how the concept of integration as the limit of a sum, introduced in Section 14.1, can be used to find volumes of solids formed when curves are rotated around the x or y axis.

Prerequisites

Before starting this Section you should ...

- be able to calculate definite integrals
- understand integration as the limit of a sum

Learning Outcomes

On completion you should be able to ...

- calculate volumes of revolution

1. Volumes generated by rotating curves about the x-axis

Figure 8 shows a graph of the function $y = 2x$ for x between 0 and 3.

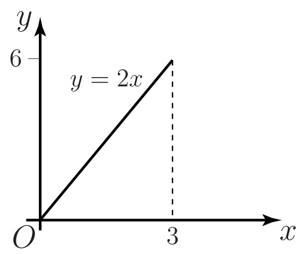


Figure 8: A graph of the function $y = 2x$, for $0 \leq x \leq 3$

Imagine rotating the line $y = 2x$ by one complete revolution (360^0 or 2π radians) around the x-axis. The surface so formed is the surface of a cone as shown in Figure 9. Such a three-dimensional shape is known as a **solid of revolution**. We now discuss how to obtain the volumes of such solids of revolution.

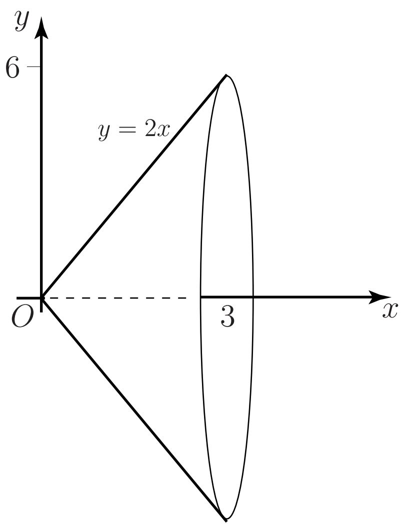
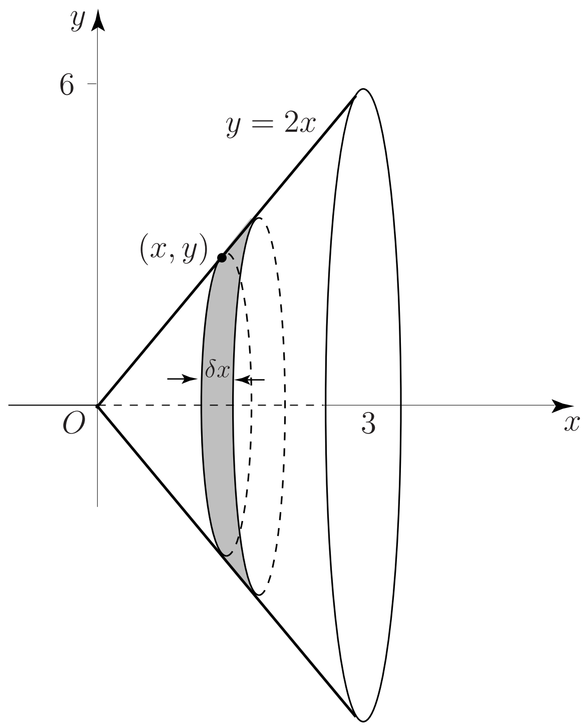


Figure 9: When the line $y = 2x$ is rotated around the axis, a solid is generated

Task

Find the volume of the cone generated by rotating $y = 2x$, for $0 \leq x \leq 3$, around the x-axis, as shown in Figure 9.

In order to find the volume of this solid we assume that it is composed of lots of thin circular discs all aligned perpendicular to the x-axis, such as that shown in the diagram. From the diagram below we note that a typical disc has radius y, which in this case equals $2x$, and thickness δx.



The cone is divided into a number of thin circular discs.

The volume of a circular disc is the circular area multiplied by the thickness.

Write down an expression for the volume of this typical disc:

Your solution

Answer

$\pi(2x)^2\delta x = 4\pi x^2 \delta x$

To find the total volume we must sum the contributions from all discs and find the limit of this sum as the number of discs tends to infinity and δx tends to zero. That is

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=3} 4\pi x^2 \delta x$$

This is the definition of a definite integral. Write down the corresponding integral:

Your solution

Answer

$\displaystyle\int_0^3 4\pi x^2\, dx$

Find the required volume by performing the integration:

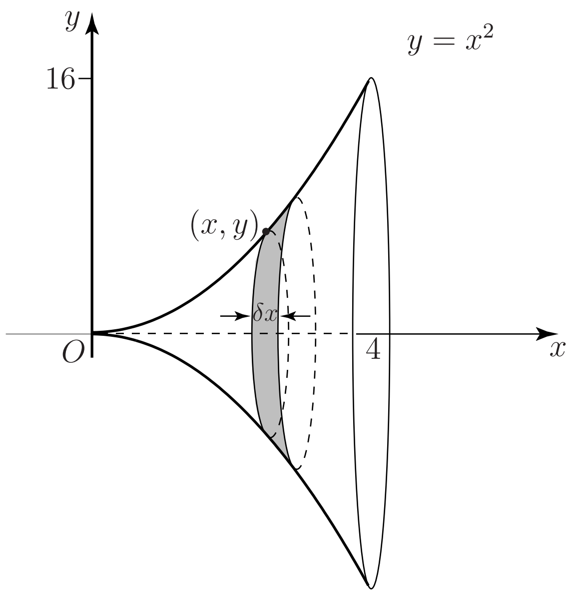
Your solution

Answer

$$\left[\frac{4\pi x^3}{3}\right]_0^3 = 36\pi$$

Task

A graph of the function $y = x^2$ for x between 0 and 4 is shown in the diagram. The graph is rotated around the x-axis to produce the solid shown. Find its volume.



The solid of revolution is divided into a number of thin circular discs.

As in the previous Task, the solid is considered to be composed of lots of circular discs of radius y, (which in this example is equal to x^2), and thickness δx.

Write down the volume of each disc:

Your solution

Answer

$\pi(x^2)^2\,\delta x = \pi x^4 \delta x$

Write down the expression which represents summing the volumes of all such discs:

Your solution

Answer

$$\sum_{x=0}^{x=4} \pi x^4 \delta x$$

Write down the integral which results from taking the limit of the sum as $\delta x \to 0$:

Your solution

Answer

$\int_0^4 \pi x^4\,dx$

Perform the integration to find the volume of the solid:

Your solution

Answer

$\dfrac{4^5\pi}{5} = 204.8\pi$

Task

In general, suppose the graph of $y(x)$ between $x = a$ and $x = b$ is rotated about the x-axis, and the solid so formed is considered to be composed of lots of circular discs of thickness δx.

Write down an expression for the radius of a typical disc:

Your solution

Answer

y

Write down an expression for the volume of a typical disc:

Your solution

Answer

$\pi y^2 \delta x$

The total volume is found by summing these individual volumes and taking the limit as δx tends to zero:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x$$

Write down the definite integral which this sum defines:

Your solution

Answer

$\int_a^b \pi y^2\,dx$

Key Point 5

If the graph of $y(x)$, between $x = a$ and $x = b$, is rotated about the x-axis the volume of the solid formed is

$$\int_a^b \pi y^2\, dx$$

Exercises

1. Find the volume of the solid formed when that part of the curve between $y = x^2$ between $x = 1$ and $x = 2$ is rotated about the x-axis.
2. The parabola $y^2 = 4x$ for $0 \leq x \leq 1$, is rotated around the x-axis. Find the volume of the solid formed.

Answers 1. $31\pi/5$, 2. 2π.

2. Volumes generated by rotating curves about the y-axis

We can obtain a different solid of revolution by rotating a curve around the y-axis instead of around the x-axis. See Figure 10.

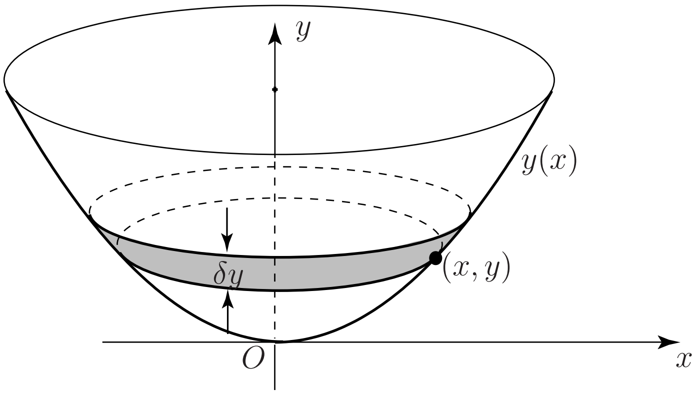


Figure 10: A solid generated by rotation around the y-axis

To find the volume of this solid it is divided into a number of circular discs as before, but this time the discs are horizontal. The radius of a typical disc is x and its thickness is δy. The volume of the disc will be $\pi x^2 \delta y$.

The total volume is found by summing these individual volumes and taking the limit as $\delta y \to 0$. If the lower and upper limits on y are c and d, we obtain for the volume:

$$\lim_{\delta y \to 0} \sum_{y=c}^{y=d} \pi x^2 \delta y \quad \text{which is the definite integral} \quad \int_c^d \pi x^2\, dy$$

Key Point 6

If the graph of $y(x)$, between $y = c$ and $y = d$, is rotated about the y-axis the volume of the solid formed is

$$\int_c^d \pi x^2 \, dy$$

Task

Find the volume generated when the graph of $y = x^2$ between $x = 0$ and $x = 1$ is rotated around the y-axis.

Using Key Point 6 write down the required integral:

Your solution

Answer

$$\int_0^1 \pi x^2 \, dy$$

This integral can be written entirely in terms of y, using the fact that $y = x^2$ to eliminate x. Do this now, and then evaluate the integral:

Your solution

Answer

$$\int_0^1 \pi x^2 \, dy = \int_0^1 \pi y \, dy = \left[\frac{\pi y^2}{2}\right]_0^1 = \frac{\pi}{2}$$

Exercises

1. The curve $y = x^2$ for $1 < x < 2$ is rotated about the y-axis. Find the volume of the solid formed.

2. The line $y = 2 - 2x$ for $0 \le x \le 2$ is rotated around the y-axis. Find the volume of revolution.

Answers

1. $\dfrac{15\pi}{2}$ 2. $\dfrac{16\pi}{3}$.

Lengths of Curves and Surfaces of Revolution 14.4

Introduction

Integration can be used to find the length of a curve and the area of the surface generated when a curve is rotated around an axis. In this Section we state and use formulae for doing this.

Prerequisites Before starting this Section you should ...	• be able to calculate definite integrals
Learning Outcomes On completion you should be able to ...	• find the length of curves • find the area of the surface generated when a curve is rotated about an axis

1. The length of a curve

To find the length of a curve in the xy plane we first divide the curve into a large number of pieces. We measure (or, at least, approximate) the length of each piece and then by an obvious summation process obtain an estimate for the length of the curve. Theoretically, we allow the number of pieces to increase without bound, implying that the length of each piece will tend to zero. In this limit the summation process becomes an integration process.

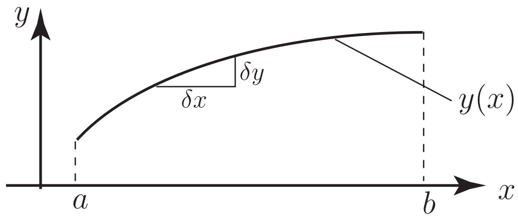


Figure 11

Figure 11 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$. A small piece of this curve has been selected and can be considered as the hypotenuse of a triangle with base δx and height δy. (Here δx and δy are intended to be 'small' so that the **curved segment** can be regarded as a **straight segment**.)

Using Pythagoras' theorem, the length of the hypotenuse is: $\sqrt{\delta x^2 + \delta y^2} = \sqrt{1 + \left(\dfrac{\delta y}{\delta x}\right)^2}\ \delta x$

By summing all such contributions between $x = a$ and $x = b$, and letting $\delta x \to 0$ we obtain an expression for the total length of the curve:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\ \delta x$$

But we already know how to write such an expression in terms of an integral. We obtain the following result:

Key Point 7

Given a curve with equation $y = f(x)$, then the length of the curve between the points where $x = a$ and $x = b$ is given by the formula:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Because of the complicated form of the integrand, and in particular the square root, integrals of this type are often difficult to calculate. In practice, approximate numerical methods rather than exact methods are normally needed to perform the integration. We shall first illustrate the application of the formula in Key Point 7 by a problem which could be calculated in a much simpler way, before looking at some harder problems.

Example 5

Find the length of the curve $y = 3x + 2$ between $x = 1$ and $x = 5$.

Solution

In this Example, the curve is in fact a straight line segment, and its length could be obtained using Pythagoras' theorem without the need for integration.

Notice from the formula in Key Point 7 that it is necessary to find $\dfrac{dy}{dx}$, which in this case is 3.

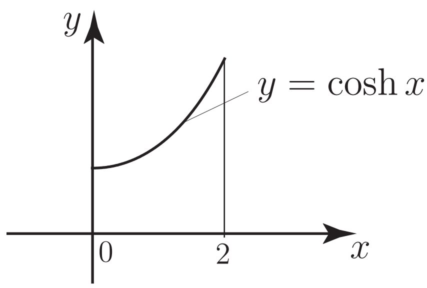
Applying the formula we find

$$\begin{aligned}
\text{length of curve} &= \int_1^5 \sqrt{1+(3)^2}\,dx \\
&= \int_1^5 \sqrt{10}\,dx \\
&= \Big[\sqrt{10}x\Big]_1^5 \\
&= (5-1)\sqrt{10} = 4\sqrt{10} = 12.65 \quad \text{to 2 d.p.}
\end{aligned}$$

Thus the length of the curve $y = 3x + 2$ between the points where $x = 1$ and $x = 5$ is 12.65 units.

Task

Find the length of the curve $y = \cosh x$ between $x = 0$ and $x = 2$ shown in the diagram.



First write down $\dfrac{dy}{dx}$:

Your solution

$\dfrac{dy}{dx} =$

Answer

$\dfrac{dy}{dx} = \sinh x$

Hence write down the required integral:

Your solution

Answer
$$\int_0^2 \sqrt{1+\sinh^2 x}\,dx$$

This integral can be evaluated by making use of the hyperbolic identity $\cosh^2 x - \sinh^2 x \equiv 1$.

Write down the integral which results after applying this identity:

Your solution

Answer
$$\int_0^2 \cosh x\,dx$$

Perform the integration to find the required length:

Your solution

Answer
$$\Big[\sinh x\Big]_0^2 = 3.63 \text{ to 2 d.p.}$$

Thus the length of $y = \cosh x$ between $x = 0$ and $x = 2$ is 3.63 units.

The next Task is more complicated still and requires the use of a hyperbolic substitution and knowledge of the hyperbolic identities.

Task

Find the length of the curve $y = x^2$ between $x = 0$ and $x = 3$.

Given $y = x^2$ then $\dfrac{dy}{dx} = 2x$. Use this result and apply the formula in Key Point 7 to obtain the integral required:

Your solution

Answer
$$\int_0^3 \sqrt{1+4x^2}\,dx$$

Make the substitution $x = \frac{1}{2}\sinh u$, giving $\dfrac{dx}{du} = \dfrac{1}{2}\cosh u$, to obtain an integral in terms of u:

Your solution

Answer

$$\int_0^{\sinh^{-1} 6} \sqrt{1+\sinh^2 u}\,\frac{1}{2}\cosh u\, du$$

Use the hyperbolic identity $\cosh^2 u - \sinh^2 u \equiv 1$ to eliminate $\sinh^2 u$:

Your solution

Answer

$$\frac{1}{2}\int_0^{\sinh^{-1} 6} \cosh^2 u\, du$$

Use the hyperbolic identity $\cosh^2 u \equiv \frac{1}{2}(\cosh 2u + 1)$ to rewrite the integrand in terms of $\cosh 2u$:

Your solution

Answer

$$\frac{1}{4}\int_0^{\sinh^{-1} 6} (\cosh 2u + 1)\, du$$

Finally, perform the integration to complete the calculation:

Your solution

Answer

$$\begin{aligned}\frac{1}{4}\int_0^{\sinh^{-1} 6} (\cosh 2u + 1)\, du &= \frac{1}{4}\left[\frac{\sinh 2u}{2} + u\right]_0^{\sinh^{-1} 6} \\ &= 9.75 \quad \text{to 2 d.p.}\end{aligned}$$

Thus the length of the curve $y = x^2$ between $x = 0$ and $x = 3$ is 9.75 units.

Exercises

1. Find the length of the line $y = 2x + 7$ between $x = 1$ and $x = 3$ using the technique of this Section. Verify your result from your knowledge of the straight line.

2. Find the length of $y = x^{3/2}$ between $x = 0$ and $x = 5$.

3. Calculate the length of the curve $y^2 = 4x^3$ between $x = 0$ and $x = 2$, in the first quadrant.

Answers

1. $2\sqrt{5} \approx 4.47$. The distance is from (1.9) to $(3, 13)$ along the line. This is given using Pythagoras' theorem as $\sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$.

2. 12.41

3. 6.06 (first quadrant only).

2. The area of a surface of revolution

In Section 14.2 we found an expression for the volume of a solid of revolution. Here we consider the more complicated problem of formulating an expression for the surface area of a solid of revolution.

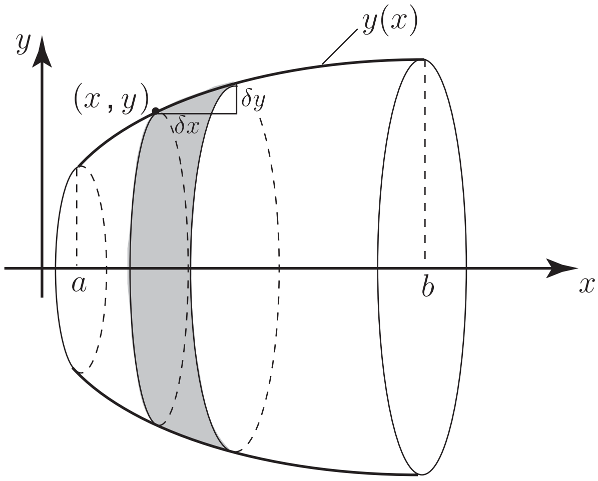


Figure 12

Figure 12 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$ which is rotated around the x axis through $360°$. A small disc, of thickness δx, of the solid of revolution has been selected. Its radius is y and so its circumference has length $2\pi y$. (As usual we assume δx is 'small' so that the **curved** part of $y(x)$ representing the hypotenuse of the highlighted 'triangle' can be regarded as **straight**). This surface 'ribbon', shown shaded, has a length $2\pi y$ and a width $\sqrt{(\delta x)^2 + (\delta y)^2}$ and so its area is, to a good approximation, $2\pi y\sqrt{(\delta x)^2 + (\delta y)^2}$. We now let $\delta x \to 0$ to obtain the result in Key Point 8:

Key Point 8

Given a curve with equation $y = f(x)$, then the surface area of the solid generated by rotating that part of the curve between the points where $x = a$ and $x = b$ around the x axis is given by the formula:

$$\text{area of surface} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Task

Find the area of the surface generated when the part of the curve $y = x^3$ between $x = 0$ and $x = 4$ is rotated around the x axis.

Using Key Point 8 write down the integral:

Your solution

Answer

$$\text{area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 2\pi x^3 \sqrt{1 + (3x^2)^2}\, dx = \int_0^4 2\pi x^3 \sqrt{1 + 9x^4}\, dx$$

Use the substitution $u = 1 + 9x^4$ so $\dfrac{du}{dx} = 36x^3$ to write down the integral in terms of u:

Your solution

Answer

$$\frac{\pi}{18} \int_1^{2305} \sqrt{u}\, du$$

Perform the integration:

Your solution

Answer

$$\frac{\pi}{18} \left[\frac{2u^{3/2}}{3}\right]_1^{2305}$$

Apply the limits of integration to find the area:

Your solution

Answer

$$\frac{\pi}{27}\left((2305)^{3/2} - 1\right)$$

Exercises

1. The line $y = x$ between $x = 0$ and $x = 1$ is rotated around the x axis.
   (a) Find the area of the surface generated.
   (b) Verify this result by finding the curved surface area of the corresponding cone. (The curved surface area of a cone of radius r and slant height ℓ is $\pi r \ell$.)
2. Find the area of the surface generated when $y = \sqrt{x}$ in the interval $1 \leq x \leq 2$ is rotated about the x axis.

Answers

1. $\pi\sqrt{2}$
2. 8.28